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Nonlinear vibration of orthotropic shallow shells of the complex shape with variable thickness

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Abstract: Early R-functions theory [1] combined with variational methods have been applied to linear [2] and nonlinear vibration problems [3,4] of the shallow shells theory of the constant thickness. In the present study, we first apply R-functions theory in order to investigate the geometrically nonlinear vibrations of orthotropic shallow shells of complex shape with variable thickness. Mathematical formulation is made in the framework of classical geometrically nonlinear theory of thin shallow shells. For a discretization of the original system in time, approximation of unknown functions is carried out by using a single mode approach. In order to construct a system of basic functions, the proposed algorithm includes sequence of the linear problems such as finding eigen functions of the linear vibrations of shallow shells with variable thickness and auxiliary tasks of the elasticity theory. The linear problems are solved by the R-functions method. The developed approach allows reducing the original problem to the corresponding problem of solving nonlinear ordinary differential equations (ODEs), whose coefficients are presented in analytical form. In order to solve the obtained system of ODEs the Bubnov-Galerkin method is applied.

The proposed algorithm is implemented within an automated system POLE-RL [1]. Numerical examples of large-amplitude flexible vibrations of shallow orthotropic shells with complex shape and variable thickness are introduced demonstrating merits and advantages of the R-functions method. Comparison of the obtained results regarding shells with rectangular plans with the other methods confirms the reliability of the proposed method.

1. Introduction

Free nonlinear vibrations of open shallow shells having the constant thickness and with rectangular plan-form have been studied by a number of researches and are well documented and reviewed. However, this problem has received rather less attention for orthotropic shallow shells with complex shapes and variable thickness. A reason is mainly motivated by the complexities involved. We are going to deal with this problem the approach reported in paper [4] and referred further as RFM, which is based on application of both variational method and R-functions theory.

2. Mathematical formulation

The present formulation of the problem is based on the classical shell theory which adopts Kirchhoff's hypothesis (the rotary inertia phenomenon is not taken into account). According to this theory the non-linear strain-displacement relations at the mid-surface can be written as follows

$$\varepsilon_{11} = u,_x + \frac{w}{R_x} + \frac{1}{2} w,_x^2 ,\ \varepsilon_{22} = v,_y + \frac{w}{R_y} + \frac{1}{2} w,_y^2 ,\ \varepsilon_{12} = u,_y + v,_x + w,_x\, w,_y , \tag{1}$$

$$\chi_{11} = -\frac{\partial^2 w}{\partial x^2} ,\ \chi_{22} = -\frac{\partial^2 w}{\partial y^2} ,\ \chi_{12} = -2\frac{\partial^2 w}{\partial x \partial y} . \tag{2}$$

In these expressions, the subscripts following comma stand for partial differentiation. The constitutive relations of the shell can be expressed as follows:

$$\begin{Bmatrix} N \\ M \end{Bmatrix} = \begin{bmatrix} [C] & [0] \\ [0] & [D] \end{bmatrix} \begin{Bmatrix} \varepsilon \\ \chi \end{Bmatrix} , \tag{3}$$

where

$$[C] = \begin{bmatrix} C_{11} & C_{12} & C_{16} \\ C_{12} & C_{22} & C_{26} \\ C_{16} & C_{26} & C_{66} \end{bmatrix} , \quad [D] = \begin{bmatrix} D_{11} & D_{12} & D_{16} \\ D_{12} & D_{22} & D_{26} \\ D_{16} & D_{26} & D_{66} \end{bmatrix} . \tag{4}$$

Here C_{ij}, D_{ij} are the stiffness coefficients of the shell depending of *x* and *y*, if the shell has variable thickness.

The equation of equilibrium for free geometrically nonlinear vibration of the shallow shells may be written in the following form

$$L_{11}u + L_{12}v + L_{13}w = -Nl_1 w + m_1 \frac{\partial^2 u}{\partial t^2} , \tag{5}$$

$$L_{21}u + L_{22}v + L_{23}w = -Nl_2 w + m_1 \frac{\partial^2 v}{\partial t^2} \tag{6}$$

$$L_{31}u + L_{32}v + L_{33}w = -Nl_3 + m_1 \frac{\partial^2 w}{\partial t^2} , \tag{7}$$

where *u,v,w* are displacements of the shell in directions *Ox, Oy* and *Oz,* respectively. In equations (5)-(7) the differential operators $L_{ij}, Nl_i \;\; i, j = 1,2,3$ are defined in the similar way as in [5].

The system of equations is supplemented by boundary conditions, the expressions of which are determined by the way of shell fixation.

3. Method of solution

Applying the approach developed in references [4], we first solve the linear vibration problem and then we find the eigenfunctions. Let us denote the natural frequency and the corresponding eigenfunctions by ω_L and $w^{(c)}, u^{(c)}, v^{(c)}$, respectively. Then displacements of the non-linear problem follow

$$w(x,y,t) = y(t)w^{(c)}(x,y)\,, \tag{8}$$

$$u(x,y,t) = \Delta\big(y(t)v^{(c)}(x,y)\big) + y^2(t)u_{11}(x,y)\,,\; v(x,y,t) = \Delta\big(y(t)v^{(c)}(x,y)\big) + y^2(t)v_{11}(x,y)\,, \tag{9}$$

where functions $u_{11}(x,y), v_{11}(x,y)$ are solutions of the system being similar to the Lame's system

$$L_{11}u_{11} + L_{12}v_{11} = Nl_1\left(w^c\right), \qquad L_{21}u_{11} + L_{22}v_{11} = Nl_2\left(w^c\right). \tag{10}$$

Observe that equations (5)-(6) are identically satisfied. Symbol Δ in equations (9) is equal to 1 for shells, and zero for plates. The above mentioned problem is solved by RFM. In detail this method has been described in references [1-4].

Substituting expressions (8), (9) for $u(x,y,t)$, $v(x,y,t)$, $w(x,y,t)$ into equation (7) and applying the Bubnov-Galerkin procedure we obtain the following equation

$$\xi^{//}(\tau) + \xi(\tau) + \beta\xi^2(\tau) + \gamma\xi^3(\tau) = 0\,, \tag{11}$$

where $\tau = \omega_L t$, $\xi = \dfrac{y(t)}{h}$,

$$\beta = \frac{-1}{\omega_{L1}^2 \cdot \|w\|^2\, m_1} \iint_\Omega \Big(L_{31}u_{11} + L_{32}v_{11} - k_1 N_{11}^{(N)}\left(w^{(c)}\right) - k_2 N_{22}^{(N)}\left(w^{(c)}\right) +$$

$$+ \Delta\left(N_{11}^{(L)}\left(u^{(c)}, v^{(c)}, w^{(c)}\right)\frac{\partial^2 w^{(c)}}{\partial x^2} + 2N_{12}^{(L)}\left(u^{(c)}, v^{(c)}, w^{(c)}\right)\frac{\partial^2 w^{(c)}}{\partial x \partial y} + \right.$$

$$\left. + N_{22}^{(L)}\left(u^{(c)}, v^{(c)}, w^{(c)}\right)\frac{\partial^2 w^{(c)}}{\partial y^2} \right)\Bigg) w^{(c)} d\Omega \tag{12}$$

$$\gamma = -\frac{1}{\omega_{L1}^2 \cdot \|w\|^2\, m_1} \iint_\Omega \left(N_{11}^{(Np)}\left(u_{11}, v_{11}, w^{(c)}\right)\frac{\partial^2 w^{(c)}}{\partial x^2} + N_{22}^{(Np)}\left(u_{11}, v_{11}, w^{(c)}\right)\frac{\partial^2 w^{(c)}}{\partial y^2} + \right.$$

$$\left. + 2N_{12}^{(Np)}\left(u_{11}, v_{11}, w^{(c)}\right)\frac{\partial^2 w^{(c)}}{\partial x \partial y} \right) w^{(c)} d\Omega\,. \tag{13}$$

In formulas (12)-(13) the expressions $N_{ij}^{(N)}, N_{ij}^{(L)}, N_{ij}^{(Np)}$ are components of the following vectors

$$\vec{N}^{(N)}\left(N_{11}^{(N)}, N_{12}^{(N)} N_{12}^{(N)}\right), \quad \vec{N}^{(L)}\left(N_{11}^{(L)}, N_{12}^{(L)} N_{12}^{(L)}\right), \quad \vec{N}^{(Np)}\left(N_{11}^{(Np)}, N_{12}^{(Np)} N_{12}^{(Np)}\right), \tag{14}$$

which are defined as follows

$$\vec{N}^{(N)} = C\vec{\varepsilon}^{(N)}, \quad \vec{N}^{(L)} = C\vec{\varepsilon}^{(L)}, \quad \vec{N}^{(N)} = C\vec{\varepsilon}^{(Np)}, \tag{15}$$

where

$$\vec{\varepsilon}^{(L)} = \begin{pmatrix} \dfrac{\partial u^{(c)}}{\partial x} + k_1 w^{(c)} \\ \dfrac{\partial v^{(c)}}{\partial y} + k_2 w^{(c)} \\ \dfrac{\partial u^{(c)}}{\partial y} + \dfrac{\partial v^{(c)}}{\partial x} \end{pmatrix}, \quad \vec{\varepsilon}^{(N)} = \begin{pmatrix} \dfrac{1}{2}\left(\dfrac{\partial w^{(c)}}{\partial x}\right)^2 \\ \dfrac{1}{2}\left(\dfrac{\partial w^{(c)}}{\partial y}\right)^2 \\ \dfrac{\partial w^{(c)}}{\partial x}\dfrac{\partial w^{(c)}}{\partial y} \end{pmatrix}, \quad \vec{\varepsilon}^{(Np)} = \begin{pmatrix} \dfrac{\partial u_{11}}{\partial x} + \dfrac{1}{2}\left(\dfrac{\partial w^{(c)}}{\partial x}\right)^2 \\ \dfrac{\partial v_{11}}{\partial y} + \dfrac{1}{2}\left(\dfrac{\partial w^{(c)}}{\partial y}\right)^2 \\ \dfrac{\partial u_{11}}{\partial y} + \dfrac{\partial v_{11}}{\partial x} + \dfrac{\partial w^{(c)}}{\partial x}\dfrac{\partial w^{(c)}}{\partial y} \end{pmatrix}. \tag{16}$$

In order to find a backbone curve we take $\xi(\tau) = A\cos\omega_N \tau$ and apply the Bubnov-Galerkin procedure [2, 3]. Then the approximate dependence between maximum amplitude A and the ratio $\nu = \omega_N / \omega_L$ [2] is as follows:

$$\nu = \sqrt{1 + \frac{8}{3\pi}\beta A + \frac{3}{4}\gamma A^2}\ . \tag{17}$$

4. Numerical results.

The correctness of the developed method have been analyzed by solving a linear vibration problem for an orthotropic clamped spherical shallow shell with the square plan-form and variable thickness of the form

$$h = h_0\left(1 + \alpha\left(6x^2 - 6x + 1\right)\right). \tag{18}$$

The material properties of the shell are

$$E_1 = 47.6\, GPa,\ E_2 = 20.7\ GPa,\ G_{12} = 5.31\ GPa,\ \nu_{12} = 0.149\,. \tag{19}$$

The comparison of obtained results with similar results reported in reference [5] confirms the reliability of the developed approach. Now, let us study the non-linear vibration of the shell with complicated form shown in Figure 1 and 2. The varying thickness is defined by (18), where as material properties are given by (19). The parameter α is varied in interval $[-0.5; 0.5]$, h_0 is thickness of the shell corresponding to $\alpha = 0$. The following geometric parameters are taken: $h_0 / a = 0.008$, $b / a = 1$, $c / a = 0.75$, $d / a = 0.6$

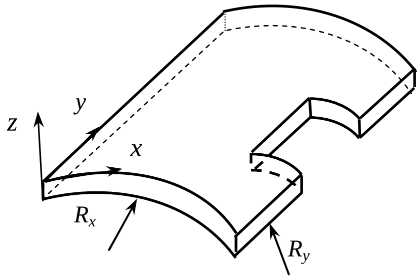

Figure 1. Shape of the shallow shell

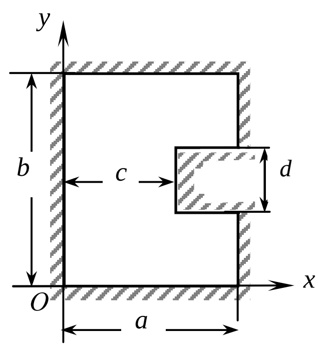

Figure 2. Plan-form of the shell

The values of non-dimensional frequencies parameter $\Lambda_i = \lambda_i (2a)^2 \sqrt{\rho h_0 / D_0}$ for the clamped orthotropic spherical shallow shell ($\frac{1}{R_x} = \frac{1}{R_y} = 0.08$) are presented in Table 1.

Table 1. Influence of variable ***thickness parameter*** α ***on frequencies*** $\Lambda_i = \lambda_i (2a)^2 \sqrt{\rho h_0 / D_0}$ *(i=1,2,3,4) of the clamped spherical shallow shell*

α	Λ_i						
	-0.5	-0.3.	-0.1	0	0.1	0.3	0.5
Λ_i	117.78	118.74	119.39	119.61	119.75	119.83	119.62
Λ_i	125.82	129.04	131.55	132.60	133.55	135.11	136.42
Λ_i	131.95	133.53	134.68	135.16	135.58	136.34	136.85
Λ_i	144.18	144.69	144.32	143.87	143.28	141.76	140.05

On the other hand we report in Figures 3 and 4 the influence of shell curvatures on backbone curves for different values of the thickness parameter α (-0.5; 0.5) obtained by our approach.

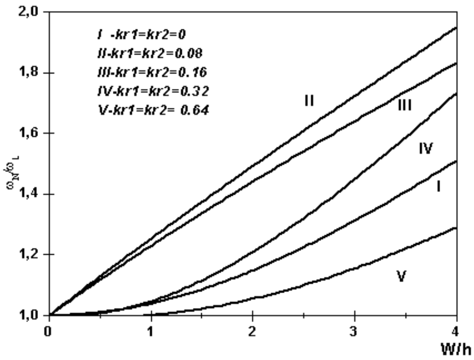

Figure 3. Effect of curvatures of the spherical shells on backbone curves $(\alpha = 0.5)$

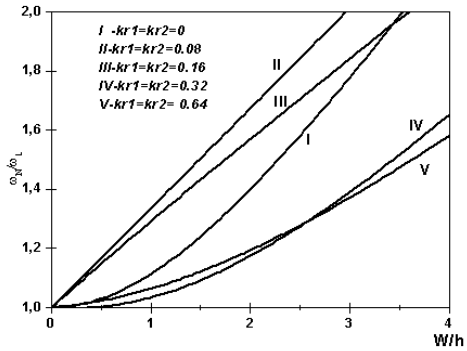

Figure 4. Effect of curvatures of the spherical shells on backbone curves $(\alpha = -0.5)$

5. Conclusions.

Analysis of geometrically non-linear vibration of the shallow shells with variable thickness and complex shapes has been carried out with a help of the R-functions theory and variational methods. Advantages of the proposed approach have been illustrated by examples including shells of variable thickness and complex shape.

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